On the Achievable Rate of ZF-DPC for MIMO Broadcast Channels with Finite Rate Feedback

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Abstract—In this paper we study a MIMO fading broadcast channel where each receiver has perfect channel state information (CSI) while the transmitter only acquires quantized CSI by finite rate feedback. We analyze the zero-forcing dirty-paper coding (ZF-DPC) scheme, which is a nonlinear precoding scheme inherently superior to linear ZF beamforming. Lower bound in closed-form expression and upper bound on the achievable ergodic rate of ZF-DPC with Gaussian inputs and uniform power allocation are derived. Based on the closed-form lower bound, sufficient conditions on the feedback channels to ensure non-zero and full downlink multiplexing gain are obtained. Specifically, in order to achieve the downlink multiplexing gain of $\alpha M$ ($0 < \alpha \leq 1$), it is sufficient to scale the number of feedback bits $B$ per user as $B = \alpha(M-1) \log_2 \frac{P}{N_0}$ where $M$ is the number of transmit antennas and $\frac{P}{N_0}$ is the average downlink SNR.

Index Terms—Broadcast channel, finite rate feedback, multiplexing gain, multiuser MIMO, zero-forcing dirty-paper coding (ZF-DPC).

I. INTRODUCTION

The multiple antenna broadcast channels, also called multiple-input multiple-output (MIMO) downlink channels, have attracted great research interest for a number of years because of their spectral efficiency improvement [1] and potential for commercial application in wireless systems. Most of existing results in this field are based on a common assumption that the transmitter in the downlink has access to perfect channel state information (CSI). In practical systems, the CSI at the transmitter (CSIT) is usually obtained by feedback from each receiver, especially for FDD systems. The quality of CSIT and so the overall system performance are directly determined by the rate of feedback.

A widely studied finite rate feedback scheme is the vector quantized CSI feedback, where each user quantizes its downlink channel coefficients using some predetermined vector quantization codebook and feeds back the bits representing the quantization index [2]-[6]. The MIMO broadcast channel with finite rate feedback has been considered in [3]-[6] where linear zero-forcing beamforming (ZF-BF) was analyzed and the results there show that the system becomes interference-limited with fixed amount of feedback while the number of feedback bits per user must increase linearly with the logarithm of the downlink SNR to maintain the full downlink multiplexing gain.

In this paper, we study the achievable rate of zero-forcing dirty-paper coding (ZF-DPC) [1], a nonlinear precoding scheme inherently superior to ZF-BF due to its interference precancellation characteristic, for a MIMO broadcast channel with finite rate feedback. We adopt the vector quantization distortion measure of the angle between the codevector and the downlink channel vector, and perform random vector quantization (RVQ) [3]-[5] for analytical convenience. Our main contributions and key findings are as follows:

- Lower bound in closed-form and upper bound on the achievable rate of ZF-DPC are derived. For fixed amount of feedback, the downlink achievable rate of ZF-DPC is bounded as the downlink SNR tends to infinity, which indicates that the downlink multiplexing gain with fixed feedback bits is zero.
- In order to achieve the downlink multiplexing gain of $\alpha M$ ($0 < \alpha \leq 1$), it is sufficient to scale the feedback bits per user as $B = \alpha(M-1) \log_2 \frac{P}{N_0}$ where $M$ is the number of transmit antennas and $\frac{P}{N_0}$ is the average downlink SNR.

We note that although ZF-DPC with limited feedback was also studied in [6], our work differs from it in several aspects. First, different distortion measures for channel vector quantization are applied, i.e., the angle between the codevector and the downlink channel vector [3]-[5] is chosen as the distortion measure in our work while the mean-square error (MSE) between the codevector and the downlink channel vector is chosen in [6]. Second, a more thorough analysis about the downlink achievable rate and multiplexing gain is presented in this paper than that in [6].

The remainder of this paper is organized as follows. We give a brief introduction to ZF-DPC with perfect CSIT in Section II. Comprehensive analysis of achievable rate and multiplexing gain are presented in Section III. Finally, conclusions are drawn in Section IV.
II. Zero-Forcing Dirty-Paper Coding with Perfect CSIT

Consider a multiple antenna broadcast channel composed of one base station (BS) with $M$ transmit antennas and $K$ users each with a single receive antenna. Assuming the channel is flat and i.i.d. block fading, the received signal at user $i$ in a given block is

$$y_i = h_i x + v_i,$$  \hspace{1cm} (1)

where $h_i \in \mathbb{C}^{1 \times M}$ is the complex channel gain vector between the BS and user $i$, $x \in \mathbb{C}^{M \times 1}$ is the transmitted signal with a total transmit power constraint $P$, i.e., $\mathbb{E}\{|x|^H x\} = P$, and $v_i$ is the complex white Gaussian noise with variance $N_0$. For analytical convenience we assume spatially independent Rayleigh fading channels between the BS and the users, i.e., the entries of $h_i$ are i.i.d. $CN(0,1)$, and $h_i$, $i = 1, \cdots, K$ are mutually independent. Under the assumption of i.i.d. block fading, $h_i$ is constant in the duration of one block and independent from block to block. By stacking the received signals of all the users into $y = [y_1 \cdots y_K]^T$, the signal model is compactly expressed as

$$y = H x + v,$$  \hspace{1cm} (2)

where $H = [h_1^T \ h_2^T \ \cdots \ h_K^T]^T$ and $v = [v_1 \ v_2 \ \cdots \ v_K]^T$.

In this paper we focus on the case $K = M$. If $K < M$, there will be a loss of multiplexing gain. The case $K > M$ will introduce multi-user diversity gain and we will leave it for future work.

Now we give a brief introduction of ZF-DPC under perfect CSIT. In the ZF-DPC scheme, the BS performs a QR-type decomposition to the overall channel matrix $H$ denoted as $H = GQ$, where $G$ is an $M \times M$ lower triangular matrix and $Q$ is an $M \times M$ unitary matrix. We let $x = Q^H d$ and the components of $d$ are generated by successive dirty-paper encoding with Gaussian codebooks [1], then the resulting signal model with the precoded transmit signal can be written as:

$$y = G d + v.$$  \hspace{1cm} (3)

From (3) the received signal at user $i$ is given by

$$y_i = g_{ii} d_i + \sum_{j<i} g_{ij} d_j + v_i,$$  \hspace{1cm} (4)

where $g_{ij} = [G]_{i,j}$ and $d_i$, the $i$-th entry of $d$, is the output of dirty-paper coding for user $i$ treating the term $\sum_{j<i} g_{ij} d_j$ as the non-causally known interference signal.

From the total transmit power constraint $\mathbb{E}\{|x|^H x\} = P$ we have $\mathbb{E}\{|d|^H d\} = P$. If the transmit power is uniformly allocated to each user, i.e., $d_i \sim CN(0, P/M)$, then for i.i.d. Rayleigh flat fading channel, the closed form expression of the achievable ergodic sum rate using ZF-DPC is given by [1], [6]:

$$R_{\text{sum}}^{\text{ZF-DPC}} = \sum_{i=1}^{M} R_i^{\text{ZF-DPC}}$$  \hspace{1cm} (5)

and

$$R_i^{\text{ZF-DPC}} = \mathbb{E}\left\{ \log_2 \left( 1 + |g_{ii}|^2 \frac{P}{MN_0} \right) \right\} = e^{\frac{MN_0}{P}} \log_2 e \sum_{j=1}^{M-i+1} E_j \left( \frac{MN_0}{P} \right),$$  \hspace{1cm} (6)

where $E_n(x) \triangleq \int_{1}^{\infty} e^{-xt} t^{-n} dt$ is the exponential integral function of order $n$ [7].

The multiplexing gain of ZF-DPC under perfect CSIT is $M$, i.e.,

$$\lim_{\frac{P}{MN_0} \to \infty} \frac{R_{\text{sum}}^{\text{ZF-DPC}}}{\log_2 e} = M,$$  \hspace{1cm} (7)

which is the full multiplexing gain of the downlink [3].

III. Achievable Rate of ZF-DPC with Finite Rate Feedback

In this section we consider the finite rate feedback scheme. The feedback channels are assumed to be error-free, i.e., the feedback bits conveyed by each user can be received by the BS correctly. We also assume perfect CSIR and no feedback delay in order to focus on the impact of finite rate feedback. For ease of analysis, we impose two restrictions on the transmission strategy: 1) the total transmit power is equally allocated to the users, and 2) independent Gaussian encoding is applied for each user at the transmitter side.

A. Finite Rate Feedback With Random Vector Quantization

The downlink channel vector $h_i$ of user $i$ can be expressed as $h_i = \lambda_i \hat{h}_i$, where $\lambda_i \triangleq |h_i|$ is the amplitude of $h_i$ and $\hat{h}_i \triangleq h_i/|h_i|$ is the direction of $h_i$. Under the assumption that the entries of $h_i$ are i.i.d. $CN(0,1)$, we have $\lambda_i^2 \sim \chi^2_M$ and $\hat{h}_i$ is uniformly distributed on the $M$ dimensional complex unit sphere [2]. Moreover, $\lambda_i$ and $\hat{h}_i$ are independent of each other [2].

The Random Vector Quantization (RVQ) [2][3] is adopted in our analysis due to its analytical tractability and close performance to the optimal quantization. The quantization codebook is randomly generated for each quantization process, and we analyze performance averaged over all such choices of random codebooks, in addition to averaging over the fading distribution. At the receiver end of user $i$, $\hat{h}_i$ is quantized using RVQ. First, a random vector codebook $C_i = \{c_{i,1}, \cdots, c_{i,N}\}$ ($N = 2^B$) is generated for user $i$ by selecting each of the $N$ vectors independently from the uniform distribution on the $M$ dimensional complex unit sphere, i.e., the same distribution as $\hat{h}_i$. The codebooks for different users are also independently generated to avoid the case that multiple users quantize their channel directions to the same quantization vector. The BS is assumed to know the codebooks generated each time by the users. Then the code vector which has the largest absolute square inner product with $\hat{h}_i$ is picked up as the quantization result, mathematically formulated as follows:

$$\hat{h}_i = \arg \max_{c \in W_i} |c \cdot \hat{h}_i|^2.$$  \hspace{1cm} (8)
The $B = \log_2 N$ quantization bits are fed back to the BS. Define $\nu_i \triangleq |\hat{h}_i \bar{H}_i^H|^2$ and $\theta_i \triangleq \angle(\hat{h}_i \bar{H}_i^H)$ where $\angle(\cdot)$ denotes the phase angle of a complex number.

In the next subsection we will find that the information of $\theta_i$ is necessary for phase compensation at user $i$’s receiver. Therefore, we need to store the value of $\theta_i$ at user $i$’s receiver. Notice that the norm information of the channel vectors are not conveyed to the BS.

**B. Lower Bound on the Achievable Rate of ZF-DPC**

The BS reconstructs the quantized channel vector $\hat{h}_i$ using the $B$ bits fed back from user $i$ and treats $\hat{h}_i$ as the true channel vector. Then the BS performs ZF-DPC using the reconstructed channel matrix $\hat{H} \triangleq [\hat{h}_1^T \cdots \hat{h}_M^T]^T$ as the true one. The QR decomposition of $\hat{H}$ can be written as $\hat{H} = \hat{G}\hat{Q}$, where $\hat{G}$ is a lower triangular matrix and $\hat{Q}$ is a unitary matrix. The received signal is modeled as:

$$y = \hat{H}\hat{Q}^Hd + v = \Lambda\hat{H}\hat{Q}^Hd + \Theta v, \quad (9)$$

where $\Lambda \triangleq \text{diag}(\lambda_1, \cdots, \lambda_M)$ is a diagonal matrix, and $\hat{H} \triangleq [\hat{h}_1^T \cdots \hat{h}_M^T]^T$.

At each user’s receiver, a phase compensation operation is carried out by multiplying $e^{j\theta_i}$ to the received signal of user $i$, written in a compact form as follows:

$$r = \Theta y = \Theta \Lambda \hat{H}\hat{Q}^Hd + \Theta v = \Lambda \Theta \hat{H}\hat{Q}^Hd + w, \quad (10)$$

where $\Theta \triangleq \text{diag}(e^{j\theta_1}, \cdots, e^{j\theta_M})$ is a diagonal matrix, $w \triangleq \Theta v$ has the same statistics as $v$.

Denote $\Delta_i \triangleq e^{j\theta_i}\hat{h}_i - \bar{h}_i$, then we can rewrite it in a compact form, i.e., $\Theta\hat{H} = \hat{H} + \Delta$, where $\Delta \triangleq [\Delta_1^T \cdots \Delta_M^T]^T$.

Equation (10) can be rewritten as:

$$r = \Lambda (\hat{H} + \Delta)\hat{Q}^Hd + w = \Lambda \hat{G}d + \Lambda \Delta \hat{Q}^Hd + w, \quad (11)$$

From the above equation we can extract the received signal at user $i$ as listed below:

$$r_i = \lambda_i \left( \hat{g}_{ii}d_i + \sum_{j<i} \hat{g}_{ij}d_j + \Delta_i \hat{Q}^Hd \right) + w_i. \quad (12)$$

We first give three lemmas useful for deriving the lower bound of the achievable rate of ZF-DPC under digital feedback.

**Lemma 1:** $|\lambda_i\hat{g}_{ii}|^2 \sim \chi^2_{2M-i+1}$.

**Lemma 2:** $E\{\Delta_i \Delta_i^H\} = 2 \left(1 - E\{\sqrt{\nu_i}\}\right)$ in which

$$E\{\sqrt{\nu_i}\} = 1 - \sum_{k=0}^{N} \binom{N}{k} (-1)^k \cdot \frac{[2k(M-1)]!!}{[2k(M-1)+1]!!}, \quad (13)$$

where $N = 2^B$, $[2k]!! \triangleq 2 \cdot 4 \cdots (2k-2) \cdot 2k$ and $[2k+1]!! \triangleq 1 \cdot 3 \cdots (2k-1) \cdot (2k+1)$.

The proofs of these two lemmas are in Appendix A and B respectively. Then we have the following theorem on the lower bound of the achievable ergodic rate of ZF-DPC under finite rate feedback.

**Theorem 1:** If the downlink channel is i.i.d. Rayleigh fading, then the achievable ergodic rate of ZF-DPC with finite rate feedback is lower bounded as:

$$R_{i}^{FB} \geq \log_2 e \cdot \psi(M - i + 1) + \log_2 \left( \frac{P}{MN_0} \right) - \log_2 \left( 1 + \frac{P}{N_0} \cdot E\{\Delta_i \Delta_i^H\} \right), \quad (14)$$

where $\psi(x)$ is the Euler psi function [7] and $E\{\Delta_i \Delta_i^H\}$ is given in Lemma 2.

**Proof:** We first consider the lower bound on the achievable rate under fixed $\hat{H}$ and $\Lambda$. Since $\lambda_i$ is known by the receiver of user $i$, the signal model in (12) can be transformed into:

$$r_i' = r_i/\lambda_i = \hat{g}_{ii}d_i + \sum_{j<i} \hat{g}_{ij}d_j + \Delta_i \hat{Q}^Hd + w_i/\lambda_i = x_i + s_i + n_i, \quad (15)$$

where $x_i = \hat{g}_{ii}d_i$, $s_i = \sum_{j<i} \hat{g}_{ij}d_j$ and $n_i = \Delta_i \hat{Q}^Hd + w_i/\lambda_i$.

With uniform power allocation among the $M$ users and independent Gaussian encoding, $d_i \sim \mathcal{CN}(0, \frac{P}{M})$, $d_i$ and $d_j$ ($i \neq j$) are independent of each other. So $x_i$ and $s_i$ are mutually independent, but $n_i$ is no longer Gaussian and is not independent of $x_i$, so we cannot directly apply the result of dirty-paper coding [9] to derive the capacity of this channel.

As $s_i$ is still known at the transmitter, from [10] we know that the achievable rate of this kind of channel can be formulated in the form of mutual information as shown below:

$$R_i^{FB}(\hat{H}, \Lambda) = I(u_i; y_i) - I(u_i; s_i) = h(u_i) - h(u_i|y_i) + h(u_i|s_i) = h(u_i|s_i), \quad (16)$$

where $u_i$ is an auxiliary random variable. Let $u_i = x_i + s_i$, then

$$R_i^{FB}(\hat{H}, \Lambda) = h(u_i - s_i|s_i) - h(u_i - y_i|y_i) = h(x_i) - h(-n_i) = h(x_i) - h(-n_i) \geq h(x_i) - \log_2(\pi e \cdot \text{Var}(-n_i)), \quad (17)$$

where the first “$\geq$” follows from the fact that the entropy is larger than the conditional entropy, and the second “$\geq$” follows from the fact that a Gaussian random variable has the largest differential entropy when the mean and variance of a random variable are given.

As $d_i \sim \mathcal{CN}(0, \frac{P}{M})$, we have $h(x_i) = \log_2(\pi e \cdot |\hat{g}_{ii}|^2 \frac{P}{M})$.

Since $E\{-n_i\} = -E\{\Delta_i \hat{Q}^Hd\} - E\{d\} - E\{w_i\}/\lambda_i = 0$, then

$$\text{Var}(-n_i) = E\{n_i^2\} = \frac{P}{M} \cdot E_{\hat{h}_i \hat{h}_i^H} \{\Delta_i \Delta_i^H\} + N_0/\lambda_i^2. \quad (18)$$
Substituting (18) into (17) we finally get the following lower bound under fixed $\hat{\mathbf{H}}$ and $\Delta$:

$$R_i^{FB}(\hat{\mathbf{H}}, \Delta) \geq \log_2 \left( 1 + \lambda_i^2 \frac{P}{\Delta M N_0} \cdot E_{\hat{\mathbf{H}}, \hat{\mathbf{n}}}(\Delta, \Delta_H^H) \right). \quad (19)$$

Based on the above results, we can derive the lower bound for the achievable ergodic rate in the Rayleigh fading downlink channel. Taking the mean of both sides of the inequality in (19) we have

$$R_i^{FB} = \mathbb{E} \left( R_i^{FB}(\hat{\mathbf{H}}, \Delta) \right) \geq \mathbb{E} \left( \log_2 \left( \lambda_i \hat{g}_i |^2 \right) \right) + \log_2 \left( \frac{P}{\Delta M N_0} \right)$$

$$- \mathbb{E}_{\hat{\mathbf{h}}, \hat{\mathbf{n}}} \left( \log_2 \left( 1 + \lambda_i^2 \frac{P}{\Delta M N_0} \cdot E_{\hat{\mathbf{h}}, \hat{\mathbf{n}}}(\Delta, \Delta_H^H) \right) \right)$$

$$\geq \mathbb{E} \left( \log_2 \left( \lambda_i \hat{g}_i |^2 \right) \right) + \log_2 \left( \frac{P}{\Delta M N_0} \right) - \log_2 \left( 1 + \frac{P}{\Delta M N_0} \cdot E_{\Delta, \Delta_H^H} \right), \quad (20)$$

where the second “≥” follows from the Jensen inequality of the concave function and the fact that $\mathbb{E}[\lambda_i^2] = M$.

From Lemma 1, we can calculate the closed form expression for $\mathbb{E} \left( \log_2 \left( \lambda_i \hat{g}_i |^2 \right) \right)$:

$$\mathbb{E} \left( \log_2 \left( \lambda_i \hat{g}_i |^2 \right) \right) = \log_2 e \int_0^\infty \ln x \cdot \frac{1}{(M - i)!} \cdot x^{M - i} \cdot e^{-x} \, dx \cdot \frac{P}{\Delta M N_0}$$

$$= \log_2 e \cdot \Gamma(M - i + 1)(\psi(M - i + 1) - \ln 1)$$

$$= \log_2 e \cdot \psi(M - i + 1), \quad (21)$$

where $\psi(x)$ is the Euler psi function [7].

Substituting (21) into (20) we finally get the conclusion.

Remark: From the above theorem we can see that decreasing $\mathbb{E} \left( \Delta, \Delta_H^H \right)$ will raise the lower bound on the achievable rate. Now we give an equation on the necessity of the phase compensation operation at each receiver. In the absence of the phase compensation, the channel error vector would be $\Delta_i = \hat{\mathbf{h}}_i - \hat{\mathbf{h}}_i$. Then

$$\mathbb{E} \left( \Delta, \Delta_H^H \right) = 2 \left( 1 - \mathbb{E} \left\{ |\tilde{\mathbf{r}}(\hat{\mathbf{h}}_i, \hat{\mathbf{n}})^H| \right\} \right)$$

$$= 2 \left( 1 - \mathbb{E} \left( \sqrt{\nu_i} \cdot \cos \theta_i \right) \right). \quad (22)$$

It has been proved in [11] that $\theta_i$ is uniformly distributed in the interval $(-\pi, \pi]$ and $\theta_i$ is independent from $\nu_i$. Then $\mathbb{E} \left\{ \cos \theta_i \right\} = 0$, and

$$\mathbb{E} \left( \Delta, \Delta_H^H \right) = 2 \left( 1 - \mathbb{E} \left( \sqrt{\nu_i} \right) \cdot \mathbb{E} \left\{ \cos \theta_i \right\} \right) = 2, \quad (23)$$

which means the same lower bound remains no matter how many bits are used to quantize $\hat{\mathbf{h}}_i$. Therefore the information of $\theta_i$ and phase compensation play an important role in the finite rate feedback scheme, which is different from the case in [3] where no phase compensation is needed.

### C. Upper Bound on the Achievable Rate of ZF-DPC

An upper bound of the achievable rate is derived by assuming a genie who can provide the encoders at the BS and the decoders at the user ends with some extra information. This upper bound is referred to as the genie-aided upper-bound.

Recall equation (15) and rewrite it as follows:

$$r_i = (\hat{g}_{ii} + \Delta_i \hat{q}_i) d_i + \sum_{j<i} \hat{g}_{ij} d_j + \sum_{m \neq i} \Delta_i \hat{q}_{im} d_m + w_i / \lambda_i$$

$$= x_i + s_i + n_i, \quad (24)$$

where $\hat{q}_i$ is the $i$th column of $\hat{Q}^H$, $x_i = (\hat{g}_{ii} + \Delta_i \hat{q}_i) d_i$, $s_i = \sum_{j<i} \hat{g}_{ij} d_j$ and $n_i = \sum_{m \neq i} \Delta_i \hat{q}_{im} d_m + w_i / \lambda_i$. We also assume there is a genie who knows the values of $\lambda_i, \Delta_i, \hat{q}_i$, and $|\Delta_i \hat{q}_{im}|$ ($\forall m \neq i$) and tells these values to the encoder and decoder for user $i$, then with i.i.d. channel inputs $d_m \sim CN(0, P / M)$ ($m = 1, \cdots, M$), $n_i$ is Gaussian distributed with zero mean and variance $\text{Var}(n_i) = \sum_{m \neq i} |\Delta_i \hat{q}_{im}|^2 P / M + N_0 / \lambda_i^2$ and is independent of $x_i$. Hence the channel for user $i$ in (24) will be recognized as a standard dirty-paper channel and its capacity is $\log_2 \left( 1 + \text{Var}(x_i) / \text{Var}(n_i) \right)$ [9]. Finally the downlink achievable ergodic rate can be upper bounded by the genie-aided upper bound as given in the following theorem:

**Theorem 2**: The achievable ergodic rate of ZF-DPC with finite rate feedback, $R^{FB}_i$ ($i = 1, 2, \cdots, M$), is bounded by a genie-aided upper-bound as follows,

$$R_i^{FB} \leq \mathbb{E} \left( \log_2 \left( 1 + \frac{\lambda^2_i \hat{g}_{ii} + \Delta_i \hat{q}_i |^2 \cdot P}{\sum_{m \neq i} |\Delta_i \hat{q}_{im}|^2 \cdot P / M + N_0} \right) \right), \quad (25)$$

Simulations are needed to calculate the upper bound given in Theorem 2.

The lower and upper bounds on the achievable ergodic sum rate obtained in Theorem 1 and 2 with fixed feedback-link capacity constraint are plotted in Fig. 1. We set $M = 4$ and calculate three groups of curves where the number of feedback bits per user, i.e., $B$, is 12, 16 and 20 respectively. Achievable rate of ZF-DPC with perfect CSI-T is also plotted. The curves in Fig. 1 reveal the ceiling effect on the achievable rate.

### D. Achievable Downlink Multiplexing Gain

The multiplexing gain of the downlink with fixed feedback bits per user is zero due to the ceiling effect of the achievable rate. In order to maintain non-zero multiplexing gain, the feedback bits per user should scale with the downlink SNR. Using the lower bound obtained in Theorem 1 we can derive the following sufficient conditions on the scaling to ensure non-zero and full multiplexing gain:

**Theorem 3**: For finite rate feedback with RVQ and error-free feedback channels, assume that the number of feedback bits per user scales according to:

$$B = \alpha(M - 1) \log_2 \frac{P}{N_0}, \quad \alpha > 0, \quad (26)$$

then we have the following conclusions:

1) A sufficient condition for achieving the downlink multiplexing gain of $\alpha_0 M$ ($0 < \alpha_0 < 1$) is that $\alpha = \alpha_0$. 

2) A sufficient condition for achieving the full downlink multiplexing gain of $M$ is that $\alpha \geq 1$.

3) If $\alpha > 1$, then
\[
\lim_{N_0 \to \infty} \left( R_i^{CSIT} - R_i^{FB} \right) = 0, \quad i = 1, 2, \ldots, M.
\] (27)

The proof of Theorem 3 is in Appendix C. Note that the same conclusion has been drawn for ZF-BF in [3]. Fig. 2 illustrates the conclusions in Theorem 3. We set $M = 4$ and $\alpha = 0.5, 1, 1.5$. The curves coincide with the analytical results in Theorem 3.

IV. CONCLUSIONS

We have investigated the performance of ZF-DPC in the multiuser MIMO downlink of a FDD system where the CSIT is obtained through quantized and finite rate feedback. Lower bound in closed-form expression and upper bound on the achievable ergodic rate of ZF-DPC with Gaussian inputs and uniform power allocation are derived. Based on the closed-form lower bound, sufficient conditions on the scaling of the feedback amount to achieve non-zero and full downlink multiplexing gain are obtained. Our primary results show that it is sufficient to scale the feedback bits per user as $B = \alpha(M - 1) \log_2 \frac{P}{N_0}$ in order to achieve the downlink multiplexing gain of $\alpha M$, where $0 < \alpha \leq 1$, $M$ is the number of transmit antennas and $\frac{P}{N_0}$ is the average downlink SNR.

APPENDIX A

PROOF OF LEMMA 1

Since we use RVQ, $\hat{h}_i$ has the same distribution as $h_i$, i.e., $\hat{h}_i \sim h_i$. Therefore $\Delta \tilde{H} \sim \Delta \mathbf{H}$, i.e., the entries of $\Delta \mathbf{H}$ are i.i.d. $CN(0, 1)$. Note that $\Delta \mathbf{H} = \Delta \mathbf{G} \mathbf{Q}$ is the QR decomposition of $\Delta \mathbf{H}$ where $\Delta \mathbf{G}$ is a lower triangular matrix and $\mathbf{Q}$ is a unitary matrix. Since $\lambda_i \hat{g}_{ii}$ is the $i$th diagonal element of $\Delta \mathbf{G}$, then from [1] we can conclude that $|\lambda_i \hat{g}_{ii}|^2 \sim \chi^2_{(M-i+1)}$.

APPENDIX B

PROOF OF LEMMA 2

From the definition of $\Delta_i$, we have
\[
\Delta_i \Delta_i^H = \left(e^{j\theta_i} \hat{h}_i - \hat{h}_i\right) \left(e^{-j\theta_i} \hat{h}_i^H - \hat{h}_i^H\right)
\]
\[
= \|\hat{h}_i\|^2 - e^{j\theta_i} \hat{h}_i \hat{h}_i^H - e^{-j\theta_i} \hat{h}_i^H \hat{h}_i + \|\hat{h}_i\|^2
\]
\[
= 2 \left(1 - \Re \left(e^{-j\theta_i} \hat{h}_i \hat{h}_i^H\right)\right), \quad (B.1)
\]
where $\Re(\cdot)$ stands for the real part of a complex number. We also have
\[
\Re \left(e^{-j\theta_i} \hat{h}_i \hat{h}_i^H\right) = \Re \left(e^{-j\theta_i} \sqrt{\nu_i} e^{j\theta_i}\right) = \sqrt{\nu_i}. \quad (B.2)
\]
Therefore the following holds:
\[
\mathbb{E} \{\Delta_i \Delta_i^H\} \approx 2 \left(1 - \mathbb{E} \left\{\sqrt{\nu_i}\right\}\right). \quad (B.3)
\]

It is pointed out in [2] that the cumulative distribution function of $\nu_i$ is
\[
F_{\nu_i}(\nu) = \left(1 - (1 - \nu)^{M-1}\right)^N, \quad \nu \in [0, 1], \quad (B.4)
\]
then we can derive the closed form expressions for $\mathbb{E} \left\{\sqrt{\nu_i}\right\}$ as follows:
\[
\mathbb{E} \{\sqrt{\nu_i}\} = \int_0^1 \sqrt{\nu} \, dF_{\nu_i}(\nu)
\]
\[
= 1 - \frac{1}{2} \int_0^1 \frac{\left(1 - (1 - \nu)^{M-1}\right)^N}{\sqrt{\nu}} \, d\nu. \quad (B.5)
\]
Let $x = 1 - \nu$, then
\[
\int_0^1 \frac{\left(1 - (1 - \nu)^{M-1}\right)^N}{\sqrt{\nu}} \, d\nu = \int_0^1 \frac{(1 - x^{M-1})^N}{\sqrt{1 - x}} \, dx. \quad (B.6)
\]
$(1 - x^{M-1})^N$ can be expanded as $(1 - x^{M-1})^N = \sum_{k=0}^{N} \binom{N}{k} (-x)^k$. Moreover, we have the following integral [7]:
\[
\int_0^1 \frac{x^m}{\sqrt{1 - x}} \, dx = 2 \cdot \frac{(2m)!!}{(2m + 1)!!} \quad \text{for integer } m \geq 0. \quad (B.7)
\]
Substituting the above results into (B.5) we finally get
\[
E\{\sqrt{\nu_i}\} = 1 - \sum_{k=0}^{N} \binom{N}{k} (-1)^k \cdot \frac{2k(M-1)!!}{[2k(M-1) + 1]!!}.
\] (B.8)

**Appendix C**

**Proof of Theorem 3**

Since \(0 \leq \nu_i \leq 1\), using the results about the average quantization error of RVQ in [3], the following holds:
\[
E\{\sqrt{\nu_i}\} \geq E\{\nu_i\} \geq 1 - 2^{-\frac{\alpha_n}{\nu_i}}.
\] (C.1)

If we set \(B = \alpha(M - 1) \log_2 \frac{P}{\nu_i} \) where \(\alpha > 0\), then
\[
\frac{P}{\nu_i} \cdot E\{\Delta_i, \Delta_i^H\} = \frac{P}{\nu_i} \cdot 2 \cdot (1 - E\{\sqrt{\nu_i}\})
\leq \frac{2P}{\nu_i} \cdot 2^{-\frac{\alpha_n}{\nu_i}} = 2 \cdot \left(\frac{P}{\nu_i}\right)^{1-\alpha} \triangleq \omega_i.
\] (C.2)

For the case that \(\alpha < 1\), \(\omega_i \to \infty\) as \(P/\nu_i \to \infty\).

Substituting (C.2) into (14) we have:
\[
\lim_{\nu_i \to \infty} \frac{R_i^{FB}}{\log_2(P/\nu_i)} \geq \lim_{\nu_i \to \infty} \log_2 \left(\frac{P}{\nu_i}\right) - \log_2 \left(1 + \omega_i\right)
= 1 - (1 - \alpha) = \alpha,
\] (C.3)

which means that at least the multiplexing gain of \(\alpha M\) can be achieved when \(\alpha < 1\).

If \(\alpha = 1\), then \(\frac{P}{\nu_i} \cdot E\{\Delta_i, \Delta_i^H\} \leq 2\). Substitute it into (14) and we have:
\[
\lim_{\nu_i \to \infty} \frac{R_i^{FB}}{\log_2 \left(\frac{P}{\nu_i}\right)} \geq 1.
\] (C.4)

Since the downlink multiplexing gain cannot exceed \(M\), we finally get:
\[
\lim_{\nu_i \to \infty} \frac{R_i^{sum}}{\log_2 \left(\frac{P}{\nu_i}\right)} = M.
\] (C.5)

For the case that \(\alpha > 1\), the following holds:
\[
\lim_{\nu_i \to \infty} \frac{P}{\nu_i} \cdot E\{\Delta_i, \Delta_i^H\} = 0,
\] (C.6)
then (C.4) and (C.5) also hold.

We also have the following results for \(E_n(x)\) [8]:
\[
E_n(x) \to \begin{cases} -\gamma - \ln x + o(1), & \text{if } n = 1, \\ \frac{1}{n^\gamma}, & \text{if } n > 1, \end{cases}
\] (C.7)
as \(x \to 0\), where \(\gamma\) is the Euler-Mascheroni constant [12].

Then with (6), (C.7), (14) and (C.6), the rate gap between \(R_i^{CSI} - R_i^{FB}\) has the following asymptotic behavior:
\[
\lim_{\nu_i \to \infty} \frac{R_i^{CSI} - R_i^{FB}}{\log_2 e \left(-\gamma + \sum_{n=2}^{M-i+1} \frac{1}{n-1} - \psi(M - i + 1)\right)}.
\] (C.8)

The Euler-Mascheroni constant \(\gamma\) can be given by series [12]:
\[
\gamma = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \ln \left(1 + \frac{1}{k}\right)\right].
\] (C.9)

We also have the series representation of the Euler psi function \(\psi(x)\) [7]:
\[
\psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln \left(1 + \frac{1}{x+k}\right)\right]
\] (C.10)
Then using (C.9) and (C.10) and after some manipulations, we can get the following equation:
\[
\lim_{\nu_i \to \infty} \frac{R_i^{CSI} - R_i^{FB}}{1} \leq 0.
\] (C.12)

On the other hand, \(R_i^{CSI} - R_i^{FB}\) is definitely a nonnegative number. So the asymptotic rate gap is zero as \(\frac{P}{\nu_i}\) goes to infinity.

**References**


